

AN ROINN OIDEACHAIS AGUS EOLAÍOCHTA

LEAVING CERTIFICATE EXAMINATION, 2002

MATHEMATICS — HIGHER LEVEL

PAPER 2 (300 marks)

MONDAY, 10 JUNE — MORNING, 9.30 TO 12.00

Attempt **FIVE** Questions from Section **A** and **ONE** Question from Section **B**.
Each question carries 50 marks.

WARNING: Marks will be lost if all necessary work is not clearly shown.

SECTION A

Answer FIVE questions from this section.

1. (a) The following parametric equations define a circle:

$$x = 4 + 3 \cos \theta, \quad y = -2 + 3 \sin \theta, \quad \text{where } \theta \in \mathbf{R}.$$

What is the Cartesian equation of the circle?

- (b) The points $a(-2, 4)$, $b(0, -10)$ and $c(6, -2)$ are the vertices of a triangle.

(i) Verify that the triangle is right-angled at c .

(ii) Hence, or otherwise, find the equation of the circle that passes through the points a , b and c .

- (c) The circle C has equation $x^2 + y^2 - 4x + 6y - 12 = 0$.

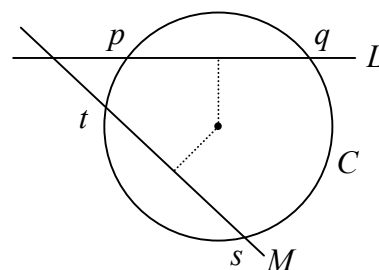
L intersects C at the points p and q .

M intersects C at the points t and s .

$$|pq| = |ts| = 8.$$

(i) Find the radius of C and hence show that the distance from the centre of C to each of the lines L and M is 3.

(ii) Given that L and M intersect at the point $(-4, 0)$, find the equations of L and M .



2. (a) $\vec{s} = 4\vec{i} - 3\vec{j}$ and $\vec{t} = 2\vec{i} - 5\vec{j}$.

Find $|\vec{st}|$.

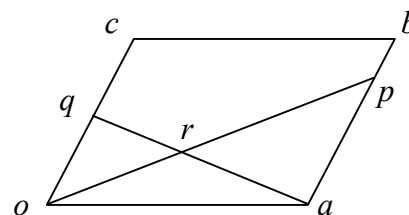
- (b) $oabc$ is a parallelogram, where o is the origin.

$p \in [ab]$ such that $|ap| : |pb| = 3 : 1$.

q is the midpoint of $[oc]$.

(i) Using equiangular triangles, or otherwise, find the ratio $|or| : |rp|$.

(ii) Express \vec{p} , and hence \vec{r} , in terms of \vec{a} and \vec{b} .



- (c) $\vec{k} = \vec{i} + 3\vec{j}$, $\vec{n} = 4\vec{i} - 2\vec{j}$, $\vec{u} = 2\vec{i} + \vec{j}$ and $\vec{v} = x\vec{i} + y\vec{j}$ where $x, y \in \mathbf{R}$.

(i) Express the value of $\vec{kn} \cdot \vec{kv}$ in the form $ax + by + c$ where $a, b, c \in \mathbf{R}$.

(ii) Prove that if $\vec{kn} \cdot \vec{kv} = \vec{kn} \cdot \vec{ku}$, and $\vec{u} \neq \vec{v}$, then $\vec{kn} \perp \vec{uv}$.

3. (a) $a(-1, 4)$ and $b(5, -4)$ are two points.
Find the equation of the perpendicular bisector of $[ab]$.
- (b) f is the transformation $(x, y) \rightarrow (x', y')$ where $x' = 3x + y$ and $y' = x - 2y$.
 S is the square whose vertices are $(0, 0)$, $(1, 0)$, $(1, 1)$ and $(0, 1)$.
- (i) Find the image under f of each of the four vertices of S .
- (ii) Express x and y in terms of x' and y' .
- (iii) By considering the lines $ax + by + c = 0$ and $ax + by + d = 0$, or otherwise, prove that f maps every pair of parallel lines to a pair of parallel lines. (You may assume that f maps every line to a line.)
- (iv) Show both S and $f(S)$ on a diagram.
- (v) Find the area of $f(S)$.

4. (a) Find the value of θ for which $\cos \theta = -\frac{\sqrt{3}}{2}$, $0^\circ \leq \theta \leq 180^\circ$.

(b) (i) Use the formula $\sin^2 A = \frac{1}{2}(1 - \cos 2A)$ to express $\sin^2 \frac{1}{2}x$ in terms of $\cos x$.

(ii) Hence, or otherwise, find all the solutions of the equation

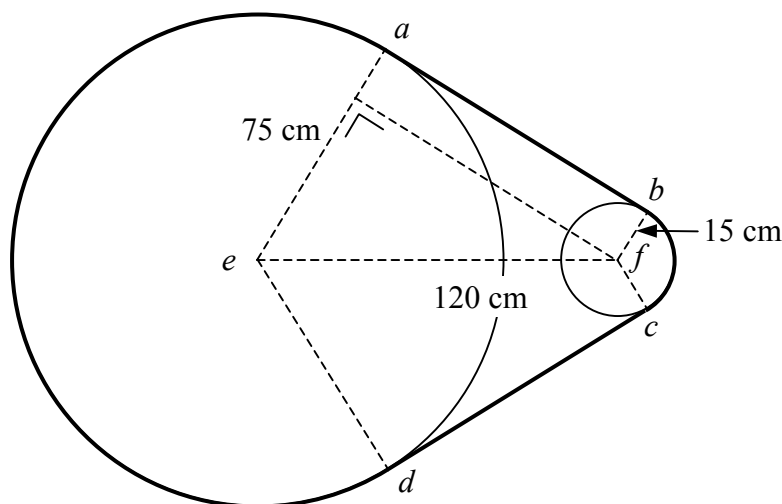
$$\sin^2 \frac{1}{2}x - \cos^2 x = 0$$

in the domain $0^\circ \leq x \leq 360^\circ$.

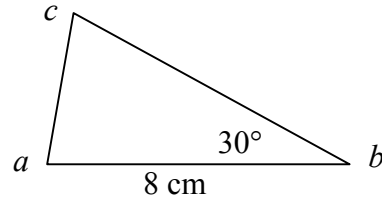
- (c) A chain passes around two circular wheels as shown. One wheel has radius 75 cm and the other has radius 15 cm. The centres, e and f , of the wheels are 120 cm apart.

The chain consists of the common tangent $[ab]$, the minor arc bc , the common tangent $[cd]$ and the major arc da .

- (i) Find the measure of $\angle aef$.
- (ii) Find $|ab|$ in surd form.
- (iii) Find the length of the chain, giving your answer in the form $k\pi + l\sqrt{3}$ where $k, l \in \mathbf{Z}$.

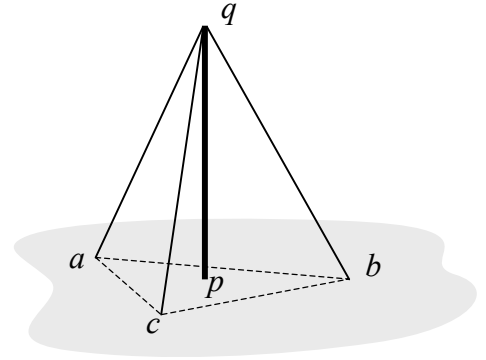


5. (a) The area of triangle abc is 12 cm^2 .
 $|ab| = 8 \text{ cm}$ and $|\angle abc| = 30^\circ$.
 Find $|bc|$.



- (b) (i) Prove that $\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$.
 (ii) Hence, or otherwise, prove that $\tan 22\frac{1}{2}^\circ = \sqrt{2} - 1$.

- (c) A vertical radio mast $[pq]$ stands on flat horizontal ground. It is supported by three cables that join the top of the mast, q , to the points a , b and c on the ground. The foot of the mast, p , lies inside the triangle abc .



Each cable is 52 m long and the mast is 48 m high.

- (i) Find the (common) distance from p to each of the points a , b and c .
 (ii) Given that $|ac| = 38 \text{ m}$ and $|ab| = 34 \text{ m}$, find $|bc|$ correct to one decimal place.

6. (a) Nine friends wish to travel in a car. Only two of them, John and Mary, have licences to drive. Only five people can fit in the car (i.e. the driver and four others).

In how many ways can the group of five people be selected if

- (i) both John and Mary are included
 (ii) either John or Mary is included, but not both?

Later, another one of the nine friends, Anne, gets a driving licence.

- (iii) The next time the journey is made, in how many ways can the group of five be chosen, given that at least one licensed driver must be included?

- (b) (i) Solve the difference equation $6u_{n+2} - 5u_{n+1} + u_n = 0$, where $n \geq 0$, given that $u_0 = 5$ and $u_1 = 2$.
 (ii) Find an expression in n for the sum of the terms $u_0 + u_1 + u_2 + \dots + u_n$. (Hint: it is the sum of two geometric series.)
 (iii) Evaluate the sum to infinity of this series (that is: $\sum_{n=0}^{\infty} u_n$).

7. (a) Two unbiased dice, each with faces numbered from 1 to 6, are thrown.
- (i) What is the probability of getting a total equal to 8?
 - (ii) What is the probability of getting a total less than 8?
- (b) The table below shows the prices of various commodities in the year 2000, as a percentage of their prices in 1999. These are called *price relatives*. (For example, the price relative for *Food, Drink & Other Goods* is 105, indicating that the cost of these items was 5% greater in 2000 than in 1999.)

The table also shows the weight assigned to each commodity. The weight represents the importance of the commodity to the average consumer.

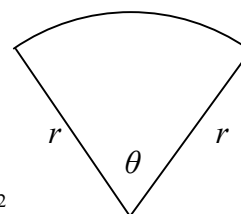
Commodity	Weight	Price in 2000 as % of price in 1999
Housing	8	110
Fuel and Transport	19	108
Tobacco	5	116
Services	16	105
Clothing & Durable Goods	10	97
Food, Drink & Other Goods	42	105

- (i) Calculate the weighted mean of the price relatives in the table.
 - (ii) Calculate, correct to two decimal places, the change in the weighted mean if *Tobacco* is removed from consideration.
- (c) A palindromic number is one that reads the same backwards as forwards, such as 727 or 38183.
- (i) This year, 2002, is a palindromic year. When is the next palindromic year?
 - (ii) How many palindromic years are there from 1000 to 9999 inclusive?
 - (iii) A whole number, greater than 9 and less than 10 000, is selected at random. What is the probability that the number is palindromic?

SECTION B

Answer ONE question from this section.

8. (a) Use integration by parts to find $\int x \ln x dx$.
- (b) The perimeter of a sector of a circle of radius r is 8 metres.
- (i) Express θ in terms of r , where θ is the angle of the sector in radians, as shown.
- (ii) Hence, show that the area of the sector, in square metres, is $4r - r^2$.
- (iii) Find the maximum possible area of the sector.



- (c) The Maclaurin series for $\tan^{-1} x$ is $x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots$

The series is convergent when $|x| < 1$.

- (i) Write down the first four terms in the series expansion for $\tan^{-1} \frac{1}{2}$.
- (ii) Use the fact that $\tan^{-1} \frac{1}{2} + \tan^{-1} \frac{1}{3} = \frac{\pi}{4}$ to derive a series expansion for π , giving the terms up to and including seventh powers.
- (iii) Use these terms to find an approximation for π . Give your answer correct to four decimal places.
9. (a) z is a random variable with standard normal distribution. Find $P(z < -0.46)$.
- (b) A certain player takes 25 penalty shots during this year's season. Each penalty shot is independent of all others. Experience from previous seasons indicates that on each occasion the probability that this player scores is $\frac{3}{5}$.
- (i) Find the probability that she scores exactly 15 of the 25 times.
- (ii) Use the normal approximation to the binomial distribution to estimate the probability that she scores at least 18 times.
- (c) (i) $P(E | F)$ denotes the conditional probability of "E given F". Write down an equation to express the relationship between $P(F)$, $P(E | F)$ and $P(E \cap F)$.
- (ii) E and F are events such that $P(E | F) = \frac{1}{2}$, $P(F | E) = \frac{1}{3}$, and $P(E \cap F) = \frac{1}{7}$. Find $P(E \cup F)$.
- (iii) Are the events E and F in part (ii) independent? Give a reason for your answer.

10. (a) The set $\{0, 2, 4, 6\}$ is a group under addition modulo 8.

Draw up its Cayley table and write down the inverse of each element.

- (b) The incomplete table shown is the Cayley table for the group $\{a, b, c, d\}, *$.

*	a	b	c	d
a	c			
b				
c			b	
d				c

(i) Explain why b must be the identity element.

(ii) Copy and complete the table.

(iii) List all of the subgroups of $\{a, b, c, d\}, *$.

- (c) (i) $G, *$ is a group and H is a non-empty subset of G .

Give a set of conditions that must be verified in order to show that $H, *$ is a subgroup of $G, *$?

(ii) G is a group and $g \in G$. Prove that the set $H = \{g^n \mid n \in \mathbf{Z}\}$ is a subgroup of G .

(iii) C is a cyclic group of order 10 and x is a generator of C . Describe all the subgroups of C in terms of x .

11. (a) The equation of an ellipse is $\frac{x^2}{25} + \frac{y^2}{9} = 1$.

Calculate the eccentricity of the ellipse.

- (b) Let f be the transformation $(x, y) \rightarrow (x', y')$, where

$$x' = 3x + 4y + 1$$

$$y' = 4x - 3y + 2.$$

Let $p(x_1, y_1)$ and $q(x_2, y_2)$ be two distinct points.

(i) Find the distance between $f(p)$ and $f(q)$ in terms of x_1, x_2, y_1 and y_2 .

(ii) Hence, or otherwise, prove that f is a similarity transformation.

- (c) $[u'v']$ is a chord of the ellipse $E: \frac{x^2}{100} + \frac{y^2}{25} = 1$.

The midpoint of $[u'v']$ is $p'(8, 2)$.

(i) Write down a linear transformation f that maps the unit circle $S: x^2 + y^2 = 1$ onto E .

(ii) Write down the co-ordinates of p , where $f(p) = p'$.

(iii) Noting that, in a circle, the line joining the centre to the midpoint of a chord is perpendicular to the chord, find the equation of uv , where $f(u) = u'$ and $f(v) = v'$.

(iv) Find the co-ordinates of u and v , and hence the co-ordinates of u' and v' .