1. (a)

$2^x = 50$

$\Rightarrow \log_{10}\left(2^x\right) = \log_{10}(50)$

$\Rightarrow x\log_{10}(2) = \log_{10}(50) \Rightarrow x = \dfrac{\log_{10}(50)}{\log_{10}(2)} = 5\cdot 64$

(b)

$2^x + 2^{-x} = 2 \Rightarrow 2^x + \dfrac{1}{2^x} = 2$

$\Rightarrow m + \dfrac{1}{m} = 2 \Rightarrow m^2 + 1 = 2m$

$\Rightarrow m^2 - 2m + 1 = 0$

$\Rightarrow (m-1)(m-1) = 0 \Rightarrow m = 1$

$\Rightarrow 2^x = 1 \Rightarrow x = 0$

(c)

$\log_6(x+1) + 2\log_6(y) = \log_6(x-1)$

$\Rightarrow 2\log_6(y) = \log_6(x-1) - \log_6(x+1)$

$\Rightarrow \log_6\left(y^2\right) = \log_6\left(\dfrac{x-1}{x+1}\right) \Rightarrow y^2 = \dfrac{x-1}{x+1}$

$\Rightarrow y = \pm\sqrt{\dfrac{x-1}{x+1}}$ but for $\log_6(y)$ to be defined $y > 0$

$x = \dfrac{5}{3} \Rightarrow y = \sqrt{\dfrac{\frac{5}{3}-1}{\frac{5}{3}+1}} = \sqrt{\dfrac{\frac{2}{3}}{\frac{8}{3}}} = \sqrt{\dfrac{2}{8}} = \sqrt{\dfrac{1}{4}} = \dfrac{1}{2}$

2. (a)

$(x+p)^2 + q \equiv x^2 - 12x + 50$

$\Rightarrow x^2 + 2px + p^2 + q \equiv x^2 - 12x + 50$

Now $x^2 + 2px + p^2 + q \equiv x^2 - 12x + 50$

$\Rightarrow 2p = -12 \Rightarrow p = -6$

and $p^2 + q = 50 \Rightarrow (-6)^2 + q = 50 \Rightarrow q = 50 - 36 = 14$

(b)

$6x^3+37x^2-34x+7=0$

$6(-7)^3+37(-7)^2-34(-7)+7=0 \Rightarrow x=-7$ is a root $\Rightarrow x+7$ is a factor.

$$
\begin{array}{r|l}
 & 6x^2-5x+1 \\
x+7 & 6x^3+37x^2-34x+7 \\
 & \underline{6x^3+42x^2} \\
 & \quad -5x^2-34x \\
 & \quad \underline{-5x^2-35x} \\
 & \qquad x+7 \\
 & \qquad \underline{x+7} \\
 & \qquad 0
\end{array}
$$

$6x^2-5x+1=(3x-1)(2x-1)=0 \Rightarrow x=\frac{1}{3}$ or $x=\frac{1}{2}$

(c)

$\log_2(x+3)=\log_2(x-1)+1$

$\Rightarrow \log_2(x+3)-\log_2(x-1)=1$

$\Rightarrow \log_2\left(\frac{x+3}{x-1}\right)=1$

$\Rightarrow \frac{x+3}{x-1}=2^1 \Rightarrow x+3=2x-2$

$2+3=2x-x \Rightarrow 5=x$

3. (a)

$\log_2(3x-1)=3$

$\Rightarrow 3x-1=2^3=8$

$\Rightarrow 3x=8+1=9 \Rightarrow x=3$

(b)

$p(x+a)^2+b \equiv 6x^2-24x+33$

$\Rightarrow p(x^2+2ax+a^2)+b \equiv 6x^2-24x+33$

$\Rightarrow px^2+2pax+pa^2+b \equiv 6x^2-24x+33$

Thus $p=6$

$2pa=-24 \Rightarrow 2\times 6\times a=-24 \Rightarrow a=-24\div 12=-2$

$pa^2+b=33 \Rightarrow 6\times(-2)^2+b=33 \Rightarrow b=33-6\times(-2)^2=9$

(c)

$x = 3$ is a root.

$(3)^3 + k(3)^2 - (3) + 21 = 0 \Rightarrow 9k + 45 = 0 \Rightarrow k = -5$

$$
\begin{array}{r l}
 & x^2 - 2x - 7 \\
x - 3 & \overline{)\, x^3 - 5x^2 - x + 21} \\
 & \underline{x^3 - 3x^2} \\
 & \quad -2x^2 - x \\
 & \quad \underline{-2x^2 + 6x} \\
 & \qquad\quad -7x + 21 \\
 & \qquad\quad \underline{-7x + 21} \\
 & \qquad\qquad 0
\end{array}
$$

$$\left.\begin{cases} ax^2 + bx + c = 0 \\ x^2 - 2x - 7 = 0 \end{cases}\right\} \quad a = 1,\ b = -2,\ c = -7$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(-7)}}{2(1)}$$

$$x = \frac{2 \pm \sqrt{32}}{2} = \frac{2 \pm \sqrt{16}\sqrt{2}}{2} = \frac{2 \pm 4\sqrt{2}}{2} = 1 \pm 2\sqrt{2}$$

4. (a)

$x^3 + 7x^2 - 4x - 28 = 0$

$(2)^3 + 7(2)^2 - 4(2) - 28 = 0 \Rightarrow x = 2$ is a root $\Rightarrow x - 2$ is a factor.

$$
\begin{array}{r l}
 & x^2 + 9x + 14 \\
x - 2 & \overline{)\, x^3 + 7x^2 - 4x - 28} \\
 & \underline{x^3 - 2x^2} \\
 & \quad 9x^2 - 4x \\
 & \quad \underline{9x^2 - 18x} \\
 & \qquad\quad 14x - 28 \\
 & \qquad\quad \underline{14x - 28} \\
 & \qquad\qquad 0
\end{array}
$$

$x^2 + 9x + 14 = (x + 2)(x + 7) = 0 \Rightarrow x = -2$ or $x = -7$

(b) (i)

$\log_2(7x + 2) - \log_2(x + 2) = 2$

$\Rightarrow \log_2\left(\frac{7x + 2}{x + 2}\right) = 2$

$\Rightarrow \frac{7x + 2}{x + 2} = 2^2 = 4 \Rightarrow 7x + 2 = 4(x + 2) = 4x + 8$

$\Rightarrow 7x - 4x = 8 - 2 \Rightarrow 3x = 6 \Rightarrow x = 2$

(ii)

$$2^x + 2^{2-x} = 4 \Rightarrow 2^x + \frac{2^2}{2^x} = 4$$

$$\Rightarrow m + \frac{4}{m} = 4 \Rightarrow m^2 + 4 = 4m$$

$$\Rightarrow m^2 - 4m + 4 = 0$$

$$\Rightarrow (m-2)(m-2) = 0 \Rightarrow m = 2$$

$$\Rightarrow 2^x = 2 \Rightarrow x = 1$$

(c)

$x^2 + ax + b$ is a factor of $x^3 + cx + d$

$$\begin{array}{r|l} & x - a \\ \hline x^2 + ax + b & \cancel{x^3} + 0x^2 + cx + d \\ & \underline{\cancel{x^3} + ax^2 + bx} \\ & \quad \cancel{-ax^2} + (c-b)x + d \\ & \quad \underline{\cancel{-ax^2} - a^2x - ab} \\ & \qquad 0 \end{array}$$

Remainder $= 0 \Rightarrow$

(i) $d = -ab \Rightarrow d + ab = 0$

(ii) $-a^2 = c - b \Rightarrow a^2 = b - c$

5. (a)

$$3^x = 1000$$

$$\Rightarrow \log_{10}(3^x) = \log_{10}(1000)$$

$$\Rightarrow x\log_{10}(3) = \log_{10}(1000) \Rightarrow x = \frac{\log_{10}(1000)}{\log_{10}(3)} = 6 \cdot 29$$

(b)

$3x + 1$ a factor $\Rightarrow x = -\frac{1}{3}$ is a root

$$\Rightarrow 3\left(-\frac{1}{3}\right)^3 + k\left(-\frac{1}{3}\right)^2 - 26\left(-\frac{1}{3}\right) - 8 = 0$$

$$\Rightarrow -\frac{1}{9} + \frac{k}{9} + \frac{26}{3} - 8 = 0 \Rightarrow \frac{k}{9} + \frac{5}{9} = 0 \Rightarrow k = -5$$

$$\begin{array}{r|l} & x^2-2x-8 \\ \hline 3x+1 & \cancel{3x^3}-5x^2-26x-8 \\ & \underline{\cancel{3x^3}+x^2} \\ & \quad \cancel{-6x^2}-26x \\ & \quad \underline{\cancel{-6x^2}-2x} \\ & \qquad \cancel{-24x-8} \\ & \qquad \underline{\cancel{-24x-8}} \\ & \qquad\quad 0 \end{array}$$

$x^2-2x-8=(x-4)(x+2)$

(c) (i)

$x^3+1000=x^3+10^3$

$\quad=(x+10)(x^2-10x+10^2)$

$\quad=(x+10)(x^2-10x+100)$

(ii)

$x^3+1000=(x+10)(x^2-10x+100)$

$\left.\begin{cases} ax^2+bx+c=0 \\ x^2-10x+100=0 \end{cases}\right\}\ a=1,\ b=-10,\ c=100$

$b^2-4ac=(-10)^2-4(1)(100)=-300<0$

thus the roots of the quadratic are unreal.

Thus $x^3+1000=(x+10)(x^2-10x+100)$ has only one real root, $x=-10$.

6. (a)

$\log_2(10x+2)-\log_2(x-1)=4$

$\log_2\left(\dfrac{10x+2}{x-1}\right)=4\Rightarrow\dfrac{10x+2}{x-1}=2^4=16$

$\Rightarrow 10x+2=16x-16\Rightarrow 16+2=16x-10x$

$\Rightarrow 18=6x\Rightarrow 3=x$

(b) (i)

Given $f(x)=ax^3+bx^2+cx+d$ and $f(k)=0$.

$\Rightarrow f(x)=f(x)-0=f(x)-f(k)$

$\quad=ax^3+bx^2+cx+d-(ak^3+bk^2+ck+d)$

$\quad=a(x^3-k^3)+b(x^2-k^2)+c(x-k)+d-d$

$\quad=a(x-k)(x^2+xk+k^2)+b(x-k)(x+k)+c(x-k)$

Thus since $(x-k)$ is a common factor for each term,

$(x-k)$ is a factor of $f(x)$.

(ii)

$g(x)=(x+1)(x-3)(x-k)$ by the factor theorem

$7g(0)+3g(2)=0$

$\Rightarrow 7(1)(-3)(-k)+3(3)(-1)(2-k)=0$

$\Rightarrow 21k-9(2-k)=0 \Rightarrow 21k-18+9k=0$

$\Rightarrow 30k-18=0 \Rightarrow k=\dfrac{18}{30}=0\cdot 6$

(c) (i)

$(x+a)^2=x^2+2ax+a^2$

$$\begin{array}{rl} & x-2a \\ x^2+2ax+a^2 & \overline{)\,\cancel{x^3}+0x^2+6px+k} \\ & \underline{\cancel{x^3}+2ax^2+a^2x} \\ & \quad \cancel{-2ax^2}+(6p-a^2)x+k \\ & \quad \underline{\cancel{-2ax^2}-4a^2x-2a^3} \\ & \qquad 0 \end{array}$$

$\text{Remainder}=0 \Rightarrow$

(i) $k=-2a^3 \Rightarrow k+2a^3=0$

(ii) $6p-a^2=-4a^2 \Rightarrow 4a^2+6p+a^2=0 \Rightarrow 3a^2+6p=0 \Rightarrow a^2+2p=0$

(iii) $k^2=(-2a^3)^2=4a^6$

$a^2+2p=0 \Rightarrow p=-\dfrac{a^2}{2} \Rightarrow 32p^3=32\left(-\dfrac{a^2}{2}\right)^3=32\left(-\dfrac{a^6}{8}\right)=-4a^6$

$\Rightarrow k^2+32p^3=4a^6-4a^6=0$