1.

$(a+b)^2 \geq 4ab$

$\Rightarrow a^2+2ab+b^2 \geq 4ab$

$\Rightarrow a^2+2ab+b^2-4ab \geq 0$

$\Rightarrow a^2-2ab+b^2 \geq 0$

$\Rightarrow (a-b)^2 \geq 0$

True, since the square of a real number cannot be negative.

2.

$x+\frac{1}{x} \geq 2$

$\Rightarrow x^2+1 \geq 2x$

(NB. If x were not positive we would have to reverse the inequality.)

$\Rightarrow x^2-2x+1 \geq 0$

$\Rightarrow (x-1)^2 \geq 0$

True, since the square of a real number cannot be negative.

3.

$x^2 \geq 3(2x-3)$

$\Rightarrow x^2 \geq 6x-9$

$\Rightarrow x^2-6x+9 \geq 0$

$\Rightarrow (x-3)^2 \geq 0$

True, since the square of a real number cannot be negative.

4.(i)

$x^2+4y^2 \geq 4xy$

$\Rightarrow x^2-4xy+4y^2 \geq 0$

$\Rightarrow (x-2y)^2 \geq 0$

True, since the square of a real number cannot be negative.

(ii)

$(x+y)^2 \le 2(x^2+y^2)$

$\Rightarrow x^2+2xy+y^2 \le 2x^2+2y^2$

$\Rightarrow 0 \le 2x^2+2y^2-x^2-2xy-y^2$

$\Rightarrow 0 \le x^2-2xy+y^2$

$\Rightarrow 0 \le (x-y)^2$

True, since the square of a real number
cannot be negative.

(iii)

$x^2+y^2-6y+9 \ge 0$

$\Rightarrow x^2+(y-3)^2 \ge 0$

True, since a sum of squares
of real numbers cannot be negative.

5(i)

$\frac{1}{a}+\frac{1}{b} \ge \frac{2}{a+b}$

$\Rightarrow \frac{b+a}{ab} \ge \frac{2}{a+b}$

$\Rightarrow (b+a)(a+b) \ge 2ab$

$\Rightarrow a^2+2ab+b^2 \ge 2ab$

$\Rightarrow a^2+b^2 \ge 0$

True, since a sum of squares
of real numbers cannot be negative

(i) Alternative

$ab(a+b)\frac{1}{a}+ab(a+b)\frac{1}{b} \ge ab(a+b)\frac{2}{a+b}$

$\Rightarrow b(a+b)+a(a+b) \ge 2ab$

$\Rightarrow ba+b^2+a^2+ab \ge 2ab$

$\Rightarrow \cancel{ba}+b^2+a^2+\cancel{ab}-\cancel{2ab} \ge 0$

$\Rightarrow b^2+a^2 \ge 0$

True, since a sum of squares
of real numbers cannot be negative

(ii)

$(a+b)\left(\frac{1}{a}+\frac{1}{b}\right) \geq 4$

$\Rightarrow (a+b)\left(\frac{b+a}{ab}\right) \geq 4$

$\Rightarrow a^2+2ab+b^2 \geq 4ab$

$\Rightarrow a^2-2ab+b^2 \geq 0$

$\Rightarrow (a-b)^2 \geq 0$

True, the square of a real number
cannot be negative

(iii)

$a+b \geq 2\sqrt{ab}$

$\Rightarrow a^2+2ab+b^2 \geq 4ab$ (Square both sides)

$\Rightarrow a^2-2ab+b^2 \geq 0$

$\Rightarrow (a-b)^2 \geq 0$

True, the square of a real number
cannot be negative

6(a)

$a^2+b^2 \geq 2ab$

$\Rightarrow a^2-2ab+b^2 \geq 0$

$\Rightarrow (a-b)^2 \geq 0$

True, the square of a real number
cannot be negative

(b)

$x^2+y^2 \geq 2xy$

$a^2+x^2 \geq 2ax$

$b^2+y^2 \geq 2by$

(c)(i)

$x^2+y^2=1 \geq 2xy$

$a^2+b^2=1 \geq 2ab$

$\Rightarrow 2 \geq 2xy+2ab$

$\Rightarrow 1 \geq xy+ab$

$\Rightarrow xy+ab \leq 1$

(ii)

$x^2+y^2=1$

$a^2+b^2=1$

$\Rightarrow 2=x^2+a^2+y^2+b^2\geq 2ax+2by$

$\Rightarrow 1\geq ax+by$

$\Rightarrow ax+by\leq 1$

7(a)

$x^2+y^2\geq 2xy$

$\Rightarrow x^2-2xy+y^2\geq 0$

$\Rightarrow (x-y)^2\geq 0$

True, the square of a real number cannot be negative

(b)

$x^2+y^2\geq 2xy$

$\Rightarrow x^4+2x^2y^2+y^4\geq 4x^2y^2$

$\Rightarrow x^4-2x^2y^2+y^4\geq 0$

$\Rightarrow \left(x^2-y^2\right)^2\geq 0$

True, the square of a real number cannot be negative

(c)

$x^4+y^4+z^4+w^4\geq 2x^2y^2+2z^2w^2$ (From part (b))

$$\begin{aligned}2x^2y^2+2z^2w^2 &= 2\left(x^2y^2+z^2w^2\right)\\ &= 2\left((xy)^2+(zw)^2\right)\\ &\geq 2\left(2(xy)(zw)\right)\end{aligned}$$ (From part (a))

Thus

$x^4+y^4+z^4+w^4\geq 4xyzw$

8(a) (i)

$f(0)=\left(0^2-1\right)(0-1)=(-1)(-1)=1\geq 0$

(ii)

$f(1)=\left(1^2-1\right)(1-1)=(0)(0)=0\geq 0$

(iii)

$f(x)=\left(x^2-1\right)(x-1)=(x+1)(x-1)(x-1)=(x+1)(x-1)^2$

Now $(x-1)^2\geq 0$ and if $x>1\Rightarrow x+1>2$

$\Rightarrow f(x)=(x+1)(x-1)^2\geq 0$ since a product of non-negative factors cannot be negative.

(iv)

$f(x)=(x^2-1)(x-1)=(x+1)(x-1)(x-1)=(x+1)(x-1)^2$

Now $(x-1)^2 \geq 0$ and if $0<x<1 \Rightarrow 1<x+1<2$

$\Rightarrow f(x)=(x+1)(x-1)^2 \geq 0$ since a product of non-negative factors cannot be negative.

(v)

$f(x)=(x^2-1)(x-1)=(x+1)(x-1)(x-1)=(x+1)(x-1)^2$

Now $(x-1)^2 \geq 0$ and if $-1<x<0 \Rightarrow 0<x+1<1$

$\Rightarrow f(x)=(x+1)(x-1)^2 \geq 0$ since a product of non-negative factors cannot be negative.

(b)

From (a) $f(x)=(x^2-1)(x-1)>0$ if $x>-1$

$\Rightarrow x^2(x-1)-1(x-1)=x^3-x^2-x+1>0$

$\Rightarrow x^3+1>x^2+x$

9.(i)

$a^3+b^3>a^2b+b^2a$

$\Rightarrow a^3+b^3-a^2b-b^2a>0$

$\Rightarrow (a+b)(a^2-ab+b^2)-ab(a+b)>0$

$\Rightarrow (a+b)(a^2-ab-ab+b^2)>0$

$\Rightarrow (a+b)(a^2-2ab+b^2)>0$

$\Rightarrow (a+b)(a-b)^2>0$

Since a and b are distinct positive numbers then $a+b$ is positive and $(a-b)^2$ is positive.

Thus, the proposition is true since the product of two positive numbers is positive.

(ii)

$a^3+c^3>a^2c+c^2a$

$b^3+c^3>b^2c+c^2b$

(iii)

$(a^2+b^2+c^2)(a+b+c)$

$=a^2(a+b+c)+b^2(a+b+c)+c^2(a+b+c)$

$=a^3+a^2b+a^2c+b^2a+b^3+b^2c+c^2a+c^2b+c^3$

$<a^3+a^3+b^3+a^3+c^3+b^3+c^3+c^3$

$\Rightarrow 3(a^3+b^3+c^3)>(a^2+b^2+c^2)(a+b+c)$

10.

$(a-b)^2$ is non-negative since it is the square of a real number.

(a^2+b^2) is non-negative since it is the sum of two squares.

Thus, $(a-b)^2(a^2+b^2)$ is non-negative since it is the product of two non-negative numbers.

$$(a-b)^2(a^2+b^2) \geq 0$$
$$\Rightarrow (a^2-2ab+b^2)(a^2+b^2) \geq 0$$
$$\Rightarrow a^2(a^2+b^2)-2ab(a^2+b^2)+b^2(a^2+b^2) \geq 0$$
$$\Rightarrow a^4+a^2b^2-2a^3b-2ab^3+b^2a^2+b^4 \geq 0$$
$$\Rightarrow a^4+b^4-2(a^3b+ab^3-b^2a^2) \geq 0$$
$$\Rightarrow a^4+b^4 \geq 2(a^3b-a^2b^2+ab^3) \text{ QED}$$