$u_n = la^n + mb^n$

We will replace a and b by numbers by solving a quadratic.
We will replace l and m by numbers, using simultaneous equations.
There will still be two letters left, u and n.

1.

$u_{n+2} - 5u_{n+1} + 4u_n = 0$

has coefficients 1, -5 and 4.

$x^2 - 5x + 4 = 0$

$\Rightarrow (x-4)(x-1) = 0 \Rightarrow x = 4$ and $x = 1$.

$u_n = la^n + mb^n$

$\Rightarrow u_n = l(4)^n + m(1)^n$

$u_0 = 3\cdot5 \Rightarrow l(4)^0 + m(1)^0 = 3\cdot5$

$\Rightarrow l + m = 3\cdot5$ A

$u_1 = 5 \Rightarrow l(4)^1 + m(1)^1 = 5$

$\Rightarrow 4l + m = 5$ B

$3l = 1\cdot5 \Rightarrow \boxed{l = 0\cdot5}$ (B − A)

Substitute in A

$0\cdot5 + m = 3\cdot5 \Rightarrow \boxed{m = 3}$

$u_n = la^n + mb^n = 0\cdot5(4)^n + 3(1)^n = 0\cdot5(4)^n + 3$

2.

$3u_{n+2}-4u_{n+1}+u_n=0$

has coefficients 3, -4 and 1.

$3x^2-4x+1=0$

$\Rightarrow(3x-1)(x-1)=0\Rightarrow x=\frac{1}{3}$ and $x=1$.

$u_n=la^n+mb^n$

$u_n=l\left(\frac{1}{3}\right)^n+m(1)^n$

$u_0=14\Rightarrow l\left(\frac{1}{3}\right)^0+m(1)^0=14$

$\Rightarrow l+m=14$ A

$u_1=8\Rightarrow l\left(\frac{1}{3}\right)^1+m(1)^1=8$

$\Rightarrow \frac{1}{3}l+m=8$ B

$\frac{2}{3}l=6\Rightarrow \boxed{l=\frac{3}{2}\times6=9}$ (A − B)

Substitute in A

$9+m=14\Rightarrow \boxed{m=5}$

$u_n=la^n+mb^n=9\left(\frac{1}{3}\right)^n+5(1)^n=9\left(\frac{1}{3}\right)^n+5$

3.

$u_{n+1}-u_n-2u_{n-1}=0$

has coefficients 1, -1 and -2.

$x^2-x-2=0$

$\Rightarrow(x-2)(x+1)=0\Rightarrow x=2$ and $x=-1$.

$u_n=la^n+mb^n$

$u_n=l(2)^n+m(-1)^n$

$u_0=4\Rightarrow l(2)^0+m(-1)^0=4$

$\Rightarrow l+m=4$ A

$u_1=-1\Rightarrow l(2)^1+m(-1)^1=-1$

$\Rightarrow 2l-m=-1$ B

$3l=3\Rightarrow \boxed{l=1}$ (A + B)

Substitute in A

$1+m=4\Rightarrow \boxed{m=3}$

$u_n=la^n+mb^n=1(2)^n+3(-1)^n=2^n+3(-1)^n$

4.

$2u_{n+2} - 11u_{n+1} + 5u_n = 0$

has coefficients 2, -11 and 5.

$2x^2 - 11x + 5 = 0$

$\Rightarrow (2x-1)(x-5) = 0 \Rightarrow x = \frac{1}{2}$ and $x = 5$.

$u_n = la^n + mb^n$

$u_n = l\left(\frac{1}{2}\right)^n + m(5)^n$

$u_0 = 2 \Rightarrow l\left(\frac{1}{2}\right)^0 + m(5)^0 = 2$

$\Rightarrow l + m = 2$ A

$u_1 = -8 \Rightarrow l\left(\frac{1}{2}\right)^1 + m(5)^1 = -8$

$\Rightarrow \frac{1}{2}l + 5m = -8$ B

$\Rightarrow -l - 10m = 16$ -2B

$-9m = 18 \Rightarrow \boxed{m = -2}$ (A $-$ 2B)

Substitute in A

$l - 2 = 2 \Rightarrow \boxed{l = 4}$

$u_n = la^n + mb^n = 4\left(\frac{1}{2}\right)^n - 2(5)^n$

5.

$6u_{n+1} = 5u_n - u_n$

$\Rightarrow 6u_{n+1} - 5u_n + u_n = 0$

has coefficients 6, -5 and 1.

$6x^2 - 5x + 1 = 0$

$\Rightarrow (3x-1)(2x-1) = 0 \Rightarrow x = \frac{1}{3}$ and $x = \frac{1}{2}$.

$u_n = la^n + mb^n$

$u_n = l\left(\frac{1}{3}\right)^n + m\left(\frac{1}{2}\right)^n$

$u_0 = 5 \Rightarrow l\left(\frac{1}{3}\right)^0 + m\left(\frac{1}{2}\right)^0 = 5$

$\Rightarrow l + m = 5$ A

$u_1 = 2 \Rightarrow l\left(\frac{1}{3}\right)^1 + m\left(\frac{1}{2}\right)^1 = 2$

$\Rightarrow \frac{1}{3}l + \frac{1}{2}m = 2 \Rightarrow -\frac{2}{3}l - m = -4$ B (multiply across by -2)

$\frac{1}{3}l = 1 \Rightarrow \boxed{l = 3}$ (A + B)

Substitute in A

$3 + m = 5 \Rightarrow \boxed{m = 2}$

$u_n = la^n + mb^n = 3\left(\frac{1}{3}\right)^n + 2\left(\frac{1}{2}\right)^n$

6. (i)

$u_{n+2} - 7u_{n+1} + 10u_n = 0$

has coefficients 1, -7 and 10.

$x^2 - 7x + 10 = 0$

$\Rightarrow (x-2)(x-5) = 0 \Rightarrow x = 2$ and $x = 5$.

$u_n = la^n + mb^n$

$u_n = l(2)^n + m(5)^n$

$u_0 = 593 \Rightarrow l(2)^0 + m(5)^0 = 593$

$\Rightarrow l + m = 593$ A

$u_1 = 1165 \Rightarrow l(2)^1 + m(5)^1 = 1165$

$\Rightarrow 2l + 5m = 1165$ B

$5l + 5m = 2965$ 5A

$-2l - 3m = -1165$ $-$B

$\Rightarrow 3l = 1800 \Rightarrow \boxed{l = 600}$ (3A $-$ B)

Substitute in A

$600 + m = 593 \Rightarrow \boxed{m = -7}$

$u_n = la^n + mb^n = 600(2)^n - 7(5)^n$

(ii)

$u_n = 600(2)^n - 7(5)^n$

$n = 2 \Rightarrow u_2 = 600(2)^2 - 7(5)^2 = 2225 > 0$

$n = 3 \Rightarrow u_3 = 600(2)^3 - 7(5)^3 = 3925 > 0$

$n = 4 \Rightarrow u_4 = 600(2)^4 - 7(5)^4 = 5255 > 0$

$n = 5 \Rightarrow u_5 = 600(2)^5 - 7(5)^5 = -2675 < 0$

Answer: $n = 5$.

7. (i)

$u_{n+2} = 9u_{n+1} - 14u_n \Rightarrow u_{n+2} - 9u_{n+1} + 14u_n = 0$

has coefficients 1, -9 and 14.

$x^2 - 9x + 14 = 0$

$\Rightarrow (x-2)(x-7) = 0 \Rightarrow x = 2$ and $x = 4$.

$u_n = la^n + mb^n$

$u_n = l(2)^n + m(7)^n$

$\boxed{u_0 = 997} \Rightarrow l(2)^0 + m(7)^0 = 997$

$\Rightarrow l + m = 997$ A

$\boxed{u_1 = 1979} \Rightarrow l(2)^1 + m(7)^1 = 1979$

$\Rightarrow 2l + 7m = 1979$ B

$7l + 7m = 6979$ 7A

$-2l - 7m = -1979$ −B

$\Rightarrow 5l = 5000 \Rightarrow \boxed{l = 1000}$ (7A − B)

Substitute in A

$1000 + m = 997 \Rightarrow \boxed{m = -3}$

$u_n = la^n + mb^n = 1000(2)^n - 3(7)^n$

(ii)

$u_2 = 1000(2)^2 - 3(7)^2 = 3853$

$u_3 = 1000(2)^3 - 3(7)^3 = 6971$

$u_4 = 1000(2)^4 - 3(7)^4 = 8797$

$u_5 = 1000(2)^5 - 3(7)^5 = -18421$

Answer: -18421

8. (i)

$4u_{n+2}-5u_{n+1}+u_n=0$

has coefficients 4, -5 and 1.

$4x^2-5x+1=0$

$\Rightarrow (4x-1)(x-1)=0 \Rightarrow x=\frac{1}{4}$ and $x=1$.

$u_n=la^n+mb^n$

$u_n=l\left(\frac{1}{4}\right)^n+m(1)^n$

$\boxed{u_0=19} \Rightarrow l\left(\frac{1}{4}\right)^0+m(1)^0=19$

$\Rightarrow l+m=19$ A

$\boxed{u_1=7} \Rightarrow l\left(\frac{1}{4}\right)^1+m(1)^1=7$

$\Rightarrow \frac{1}{4}l+m=7$ B

$\frac{3}{4}l=12 \Rightarrow \boxed{l=16}$ (A − B)

Substitute in A

$16+m=19 \Rightarrow \boxed{m=3}$

$u_n=la^n+mb^n=16\left(\frac{1}{4}\right)^n+3(1)^n=16\left(\frac{1}{4}\right)^n+3$

(ii)

$\lim_{n\to\infty} u_n=\lim_{n\to\infty}\left[16\left(\frac{1}{4}\right)^n+3\right]=16\times 0+3=3$

9.

$u_{n+2} - 2u_{n+1} - u_n = 0$

has coefficients 1, -2 and -1.

$x^2 - 2x - 1 = 0$

$$\left\{ \begin{array}{l} ax^2 + bx + c = 0 \\ x^2 - 2x - 1 = 0 \end{array} \right\} \quad a = 1,\ b = -2,\ c = -1$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(-1)}}{2(1)}$$

$$x = \frac{2 \pm \sqrt{8}}{2} = \frac{2 \pm \sqrt{4}\sqrt{2}}{2} = \frac{2 \pm 2\sqrt{11}}{2} = 1 \pm \sqrt{2}$$

$u_n = la^n + mb^n$

$u_n = l\left(1 + \sqrt{2}\right)^n + m\left(1 - \sqrt{2}\right)^n$

$\boxed{u_0 = 2} \Rightarrow l\left(1 + \sqrt{2}\right)^0 + m\left(1 - \sqrt{2}\right)^0 = 2$

$\Rightarrow l + m = 2$ A

$\boxed{u_1 = 2} \Rightarrow l\left(1 + \sqrt{2}\right)^1 + m\left(1 - \sqrt{2}\right)^1 = 2$

$\Rightarrow l\left(1 + \sqrt{2}\right) + m\left(1 - \sqrt{2}\right) = 2$ B

$\Rightarrow l + \sqrt{2}l + m - \sqrt{2}m = 2$

$\Rightarrow 2 + \sqrt{2}l - \sqrt{2}m = 2$

$\Rightarrow \sqrt{2}l - \sqrt{2}m = 0 \Rightarrow l - m = 0$ C

Add A and C: $2l = 2 \Rightarrow \boxed{l = 1}$

Substitute in A: $1 + m = 2 \Rightarrow \boxed{m = 1}$

$u_n = la^n + mb^n = 1 \times \left(1 + \sqrt{2}\right)^n + 1 \times \left(1 - \sqrt{2}\right)^n = \left(1 + \sqrt{2}\right)^n + \left(1 - \sqrt{2}\right)^n$